

# Entanglement and Extreme Spin Squeezing for a Fluctuating Number of Indistinguishable Particles

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We extend the criteria for  $k$ -particle entanglement from the spin squeezing parameter presented in [A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001)] to systems with a fluctuating number of particles. We also discuss how other spin squeezing inequalities can be generalized to this situation. Further, we give an operational meaning to the bounds for cases where the individual particles cannot be addressed. As a by-product, this allows us to show that in spin squeezing experiments with cold gases the particles are typically distinguishable in practice. Our results justify the application of the Sørensen-Mølmer bounds in recent experiments on spin squeezing in Bose-Einstein condensates.

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## I. INTRODUCTION

Spin squeezing [1–3] is a central concept in quantum metrology [4, 5] and entanglement detection [6] in systems with a large number of particles. The most prominent spin squeezing parameter, defined for  $N$  spin- $\frac{1}{2}$  particles or qubits, is [2]

$$\xi^2 = \frac{N(\Delta\hat{J}_\perp)^2}{\langle\hat{J}_\mathbf{n}\rangle^2}. \quad (1)$$

Here  $\hat{J}_\mathbf{n} = \sum_{i=1}^N \hat{j}_\mathbf{n}^{(i)}$  is a collective spin operator pointing along the direction  $\mathbf{n}$  in the Bloch sphere,  $\hat{j}_\mathbf{n}^{(i)}$  is the angular momentum operator for the particle  $i$  and  $\perp$  is a direction perpendicular to  $\mathbf{n}$ . It has been shown that a value  $\xi < 1$  implies that the state of the  $N$  particles is entangled [7]. In addition, it allows for a phase uncertainty below the shot-noise limit, *i.e.*,  $\Delta\theta < \frac{1}{\sqrt{N}}$  [2, 8, 9], when used as input of the interferometer implementing the unitary transformation  $e^{-i\theta\hat{J}_\mathbf{m}}$ , where  $\mathbf{m}$  is a direction perpendicular to both  $\mathbf{n}$  and  $\perp$ .

The relation between spin squeezing and entanglement has been further extended by Sørensen and Mølmer in Ref. [10], where bounds on  $\xi$  have been derived for a partitioning of the state into groups of at most  $1 \leq k < N$  particles. A violation of these bounds implies that there is at least one group of more than  $k$  particles that is fully entangled. Hence the state contains at least  $(k+1)$ -particle entanglement or, according to the definition in Ref. [10], an entanglement depth  $k+1$ . The criteria were applied recent experiments on spin squeezing in Bose-Einstein condensates (BECs) [11, 12]. However, while the criteria were derived for a fixed number of distinguishable atoms, the experiments were performed with a fluctuating number of bosons sharing the same trap. Hence the criteria have to be generalized to (i) a *non-fixed* number of (ii) *indistinguishable* particles.

For the case  $k = 1$ , this has been done in Ref. [13]. There, it has been shown that

$$\xi^2 = \frac{\langle\hat{N}\rangle(\Delta\hat{J}_\perp)^2}{\langle\hat{J}_\mathbf{n}\rangle^2} \quad (2)$$

is a natural generalization of the spin squeezing parameter [14]. In particular, the condition  $\xi < 1$  is sufficient for sub shot-noise phase estimation,  $\Delta\theta < \frac{1}{\sqrt{\langle\hat{N}\rangle}}$ , and signals entanglement if the input state does not contain coherences between states with a different number of particles [13]. This justifies the use of the spin squeezing parameter from Eq. (2) in experiments with cold [16–18] and ultra-cold [11, 12] atomic gases (for an exhaustive list see [3]). In Ref. [13], it has been argued that, formally, the connection between sub shot-noise sensitivity and entanglement holds also for indistinguishable particles.

In this manuscript, we extend, to the case of a fluctuating number of particles, the  $k$ -particle entanglement criteria of Ref. [10]. We also show how the generalized spin squeezing entanglement criteria of Refs [20, 21] can be extended accordingly. Afterwards, we use additional atomic degrees of freedom to extend the  $k$ -particle entanglement criteria to indistinguishable particles. We show that in a typical spin squeezing experiment with cold, but not ultra-cold atomic gases, the particles can be treated as distinguishable effectively. These results apply also to other spin squeezing criteria [3, 6, 20–30] which are generally derived for a fixed number of distinguishable particles.

The article is organized as follows. In Section II, we discuss the generalization to a nonfixed number of particles. In Section III, we consider the applicability of the bounds for indistinguishable particles, discussing explicitly cold atomic ensembles and BECs. The conclusions can be found in Section IV.

## II. SPIN-SQUEEZING BOUNDS FOR A FLUCTUATING NUMBER OF PARTICLES

Let us first recall the definition of  $(k+1)$ -particle entanglement and how it can be extended to the case of a fluctuating number of particles.

A pure state of  $N$  particles is  $k$ -producible [31, 32] if it can be written as

$$|\psi_{k\text{-prod}}^{(N)}\rangle = \otimes_{\alpha=1}^{M_N} |\psi_{\alpha}^{(N_{\alpha})}\rangle, \quad (3)$$

where  $|\psi_{\alpha}^{(N_{\alpha})}\rangle$  is a state of  $N_{\alpha} \leq k$  particles (such that  $\sum_{\alpha=1}^{M_N} N_{\alpha} = N$ ). A mixed state is  $k$ -producible if it can be written as a mixture  $\rho_{k\text{-prod}}^{(N)} = \sum_l p_l |\psi_{k_l\text{-prod}}^{(N)}\rangle \langle \psi_{k_l\text{-prod}}^{(N)}|$  with  $k_l \leq k$  for all  $l$ . A state that is  $(k+1)$ -producible but not  $k$ -producible is referred to as  $(k+1)$ -particle entangled because it contains full entanglement of at least one group of  $k+1$  particles (with  $1 \leq k < N$ ) [10, 33]. The concept of  $(k+1)$ -particle entanglement was referred to as entanglement depth in Ref. [10].

The extension of the above definition to the case of a fluctuating number of particles follows Ref. [13]. In this case, we define a quantum state to be  $k$ -producible *iff* it is  $k$ -producible in every fixed- $N$  subspace. Hence a  $k$ -producible quantum state without coherences between states of different  $N$  can be written as

$$\rho_{k\text{-prod}}^{\text{inc}} = \sum_N Q_N \rho_{k\text{-prod}}^{(N)}, \quad (4)$$

where  $\rho_{k\text{-prod}}^{(N)}$  is a state of  $N$  particles and  $\{Q_N\}$  forms a probability distribution. In practice,  $Q_N \rightarrow 0$  if  $N$  is above some threshold due to energy restrictions in the lab. For a general state  $\rho$  which may contain coherences between different  $N$ , we introduce the projection

$$\mathbb{1}_N \rho \mathbb{1}_N = Q_N \rho^{(N)}, \quad (5)$$

where  $\mathbb{1}_N$  is the projector to the subspace of  $N$  particles and  $\rho^{(N)}$  is a state on this subspace. We may then define a state to be  $k$ -producible in general if  $\rho^{(N)}$  is  $k$ -producible for any  $N$ .

Note that there is an ongoing debate about whether or not superpositions between states of different particle numbers can actually be created [34]. It turns out that since the angular momentum operator  $\hat{J}_{\mathbf{n}} = \oplus_N \hat{J}_{\mathbf{n}}^{(N)}$ , for any arbitrary direction  $\mathbf{n}$ , commutes with the number operator  $\hat{N} = \oplus_N N \mathbb{1}_N$ , such coherences do not have any effect for entanglement detection with  $\hat{J}_{\mathbf{n}}$  and its moments [23].

### A. Generalizing the Sørensen-Mølmer criteria to a fluctuating $N$

Bounds on  $\xi$  have been derived for states of  $N$  spin- $j$  particles among which at most groups of  $k$  particles are

entangled [10]. The bounds are computed with the help of the function [35]

$$F_j(X) \equiv \frac{1}{j} \min_{\rho} (\Delta \hat{j}_{\perp})^2 \Big|_{\langle \hat{j}_{\mathbf{n}} \rangle / j = X}, \quad (6)$$

where the minimization is performed over all states  $\rho$  of a spin- $j$  particle which fulfill  $\langle \hat{j}_{\mathbf{n}} \rangle / j = X$  for some  $X \in [0, 1]$  [36]. In Eq. (6),  $\hat{j}_{\perp}$  and  $\hat{j}_{\mathbf{n}}$  are spin operators for the single spin- $j$  particle. It is then shown that for  $k$ -producible states, the bound

$$(\Delta \hat{j}_{\perp})^2 \geq N j F_{kj} \left( \frac{\langle \hat{j}_{\mathbf{n}} \rangle}{N j} \right). \quad (7)$$

holds, where  $\hat{J}_{\mathbf{n}} = \sum_{i=1}^N \hat{j}_{\mathbf{n}}^{(i)}$  and in analogy for  $\hat{J}_{\perp}$ , as introduced above. Hence if the measured values of  $\langle \hat{j}_{\mathbf{n}} \rangle$  and  $(\Delta \hat{j}_{\perp})^2$  violate Eq. (7), then the state is at least  $(k+1)$ -particle entangled. For  $j = \frac{1}{2}$ , the state allows for a smaller uncertainty in an interferometric protocol than any  $k$ -producible state.

Before generalizing these bounds to states with a non-fixed  $N$ , we remark that a different method, based on the Quantum Fisher information, to detect  $(k+1)$ -particle entanglement for a state of a fixed number of particles, has been recently introduced in Ref. [37].

**Observation 1.** *For  $k$ -producible states of spin- $j$  particles with a fluctuating total number, and with given average values  $\langle \hat{N} \rangle$  and  $\langle \hat{J}_{\mathbf{n}} \rangle$ , the inequality*

$$(\Delta \hat{J}_{\perp})^2 \geq \langle \hat{N} \rangle j F_{kj} \left( \frac{\langle \hat{J}_{\mathbf{n}} \rangle}{\langle \hat{N} \rangle j} \right) \quad (8)$$

*holds, irrespective of whether or not coherences between different numbers of particles are present in the state.*

The proof is given in Appendix A. Note that Eq. (8) reduces to Eq. (7) for a fixed number of particles. Also for a fixed  $N$ , Observation 1 extends the seminal result of Ref. [10] in two ways. Firstly, in the proof, a step is carried out [below Eq. (26)] which was not discussed explicitly in the original proof. Further, Observation 1 does not require  $\frac{N}{k}$  to be an integer as in the original criterion. In order to apply it for non-fixed  $N$ , simply  $N$  has to be replaced by  $\langle \hat{N} \rangle$ , as in the usual spin squeezing criterion [13].

### B. Generalizing other spin squeezing inequalities to a fluctuating $N$

We now consider other spin squeezing inequalities for entanglement detection [3, 6, 20–30], which have been derived for a fixed number of particles. Most of them can be generalized to the case of a fluctuating number of particles by directly using the inequality

$$\begin{aligned} (\Delta \hat{J}_{\mathbf{n}})^2 &= \sum_N Q_N \langle (\hat{J}_{\mathbf{n}}^{(N)})^2 \rangle - \left( \sum_N Q_N \langle \hat{J}_{\mathbf{n}}^{(N)} \rangle \right)^2 \\ &\geq \sum_N Q_N (\Delta \hat{J}_{\mathbf{n}}^{(N)})^2, \end{aligned} \quad (9)$$

which can be derived using the Cauchy-Schwarz inequality, for any arbitrary direction  $\mathbf{n}$ . It also follows from the concavity of the variance. Further, note that for any operator  $\hat{O} = \oplus_{N=0}^{\infty} \hat{O}^{(N)}$ , which commutes with  $\hat{N}$ ,

$$\langle \hat{O}^l \rangle = \text{Tr}[\rho \hat{O}^l] = \sum_N Q_N \text{Tr}[\rho^{(N)} (\hat{O}^{(N)})^l] \quad (10)$$

holds for any power  $l$ . All angular momentum operators  $\hat{J}_{\mathbf{n}}$  are of this form. Therefore, coherences between states of different  $N$  in  $\rho$  do not play any role in entanglement detection with any kind of spin squeezing criteria, as mentioned above.

As an example, we perform the generalization for the complete set of inequalities from Ref. [20] and for the criteria detecting  $k$ -particle entanglement from Ref. [21]. All these criteria have been derived for  $N$  particles with spin  $j = \frac{1}{2}$ . The set of criteria of Ref. [20] is

$$\langle \hat{J}_{\mathbf{x}}^2 \rangle + \langle \hat{J}_{\mathbf{y}}^2 \rangle + \langle \hat{J}_{\mathbf{z}}^2 \rangle \leq N(N+2)/4 \quad (11)$$

$$(\Delta \hat{J}_{\mathbf{x}})^2 + (\Delta \hat{J}_{\mathbf{y}})^2 + (\Delta \hat{J}_{\mathbf{z}})^2 \geq N/2 \quad (12)$$

$$\langle \hat{J}_{\mathbf{i}}^2 \rangle + \langle \hat{J}_{\mathbf{j}}^2 \rangle - N/2 \leq (N-1)(\Delta \hat{J}_{\mathbf{l}})^2 \quad (13)$$

$$(N-1)[(\Delta \hat{J}_{\mathbf{i}})^2 + (\Delta \hat{J}_{\mathbf{j}})^2] \geq \langle \hat{J}_{\mathbf{l}}^2 \rangle + N(N-2)/4, \quad (14)$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{l}$  take all possible permutations of  $\mathbf{x}, \mathbf{y}, \mathbf{z}$ . This set is complete in the sense that it detects all entangled states which can be detected based on the knowledge of  $\langle \hat{J}_{\mathbf{i}}^2 \rangle$  and  $(\Delta \hat{J}_{\mathbf{i}})^2$  for  $\mathbf{i} = \mathbf{x}, \mathbf{y}, \mathbf{z}$  [20].

Due to linearity, inequality (11), which is valid for all quantum states, directly generalizes to

$$\langle \hat{J}_{\mathbf{x}}^2 \rangle + \langle \hat{J}_{\mathbf{y}}^2 \rangle + \langle \hat{J}_{\mathbf{z}}^2 \rangle \leq [\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle]/4. \quad (15)$$

Inequality (12) can be generalized using Eq. (9) to

$$(\Delta \hat{J}_{\mathbf{x}})^2 + (\Delta \hat{J}_{\mathbf{y}})^2 + (\Delta \hat{J}_{\mathbf{z}})^2 \geq \langle \hat{N} \rangle/2. \quad (16)$$

In analogy, the inequalities (13,14) can be generalized by applying Eq. (9) to the variances. The result can be written as

$$(\Delta \hat{J}_{\mathbf{l}})^2 \geq \langle (\hat{N} - 1)^{-1} \hat{J}_{\mathbf{l}}^2 \rangle \quad (17)$$

$$(\Delta \hat{J}_{\mathbf{i}})^2 + (\Delta \hat{J}_{\mathbf{j}})^2 \geq \langle (\hat{N} - 1)^{-1} \hat{J}_{\mathbf{l}}^2 \rangle + \langle (\hat{N} - 1)^{-1} \hat{J}_{\mathbf{j}}^2 \rangle - \langle (\hat{N} - 1)^{-1} \hat{N} \rangle/2 \quad (18)$$

Here it is assumed that  $Q_0 = Q_1 = 0$ . This should not pose a problem because the spin squeezing criteria are developed for a large number of particles. A conceptual change in the generalized criteria from Eqs (17) and (18) is that instead of the expectation values  $\langle \hat{J}_{\mathbf{i}}^2 \rangle$ , terms such as  $\langle (\hat{N} - 1)^{-1} \hat{J}_{\mathbf{i}}^2 \rangle$  appear. This implies that the number of particles has to be measured in each shot, which might complicate the application in some experiments. In the same way, the set of inequalities for  $N$  spin- $j$  particles from Ref. [30] can be generalized to a non-fixed  $N$ .

Note that alternatively, the criteria could be tested for a fixed number of particles  $N$ . In this case, one could collect separate statistics for each  $N$ . If the number fluctuates strongly, it would be very difficult to collect enough statistics for a given fixed  $N$ , while it is still possible to have enough statistics for the generalized criteria.

We finally remark that the bound

$$(\Delta \hat{J}_{\mathbf{z}})^2 \geq \frac{1}{k+2} \left[ \frac{\langle \hat{J}_{\mathbf{x}}^2 \rangle}{N} + \frac{\langle \hat{J}_{\mathbf{y}}^2 \rangle}{N} \right] - \frac{1}{4} \quad (19)$$

for  $k$ -producible states from Ref. [21] can be generalized to

$$(\Delta \hat{J}_{\mathbf{z}})^2 \geq \frac{1}{k+2} [\langle \hat{N}^{-1} \hat{J}_{\mathbf{x}}^2 \rangle + \langle \hat{N}^{-1} \hat{J}_{\mathbf{y}}^2 \rangle] - \frac{1}{4}. \quad (20)$$

This bound is optimal for the symmetric twin-Fock states with  $N/2$  particles in each of the two modes of an interferometer which promises a phase uncertainty close to the ultimate Heisenberg limit,  $\Delta\theta = \frac{1}{\sqrt{N}}$  [38]. Recently, such states have been prepared experimentally with ultra-cold atomic gases [39–42]. Since the number of atoms fluctuates in these experiments, Eq. (20) could be used to bound  $k$ , while Eq. (8) from Observation 1 is generally not useful in this situation since  $\langle \hat{J}_{\mathbf{z}} \rangle = 0$  for these states. However, the same problem concerning the indistinguishability of the particles occurs also here. This problem will be discussed in the next Section.

### III. SPIN-SQUEEZING BOUNDS FOR INDISTINGUISHABLE PARTICLES

The bounds (7) and (8) presented above have been derived for distinguishable particles. This corresponds to the usual situation employed in quantum information theory with, for instance, trapped ions. In this case the particles are assumed to sit at remote locations and operations are only performed on the internal degrees of freedom, locally at each trap. The particles can be treated as distinguishable, labelled by the trap number, and the (anti-)symmetrization can be dropped [44].

However, Eq. (7) has been recently applied to discuss spin squeezing experiments with Bose-Einstein condensates [11, 12]. In this situation, all the particles (bosons) share the same trap state. Their collective internal state has to be fully symmetric with respect to the interchange of any two particles in first quantization. For indistinguishable bosons, the (symmetric) fully separable states have the form  $|\phi\rangle^{\otimes N}$ . The spin squeezing condition  $\xi < 1$ , see Eqs. (1) and (2), still holds and signals entanglement in the sense that the state of the indistinguishable bosons cannot be written as  $|\phi\rangle^{\otimes N}$ . The relation between shot-noise limit and separable states holds formally as well [13, 45, 46, 48].

In contrast, a symmetric state of  $N$  particles can be either fully separable or fully entangled, but no symmetric states that are  $k$ -particle entangled as in Eq. (3) exist

for  $1 < k < N$  [50–52]. Hence the classification introduced above for distinguishable particles is not directly applicable to recent experiments with BECs, where the individual particles are not addressable. The same problem would occur if the criteria for  $k$ -particle entanglement proposed in [21] and generalized in Eq. (20) were applied to the twin-Fock states produced recently with ultra-cold atomic gases [39–42].

### A. Entanglement and spin squeezing due to symmetrization

First, let us notice that the collective spin operators  $\hat{J}_{\mathbf{n}}$ , which appears in the definition of the spin squeezing parameter  $\xi$  [Eqs. (1) and (2)], are permutationally invariant, *i.e.*,  $\hat{P}_{\pi}^{\dagger} \hat{J}_{\mathbf{n}} \hat{P}_{\pi} = \hat{J}_{\mathbf{n}}$  for any of the  $N!$  permutations  $\pi$  of the  $N$  particles (represented by  $\hat{P}_{\pi}$ ). Therefore,  $\text{Tr}[\rho \hat{J}_{\mathbf{n}}] = \text{Tr}[\rho_{\text{PI}} \hat{J}_{\mathbf{n}}]$ , where  $\rho_{\text{PI}} = \frac{1}{N!} \sum_{\pi} \hat{P}_{\pi}^{\dagger} \rho \hat{P}_{\pi}$  is permutationally invariant. One may think that, because of this property of the collective spin operators, the spin squeezing bounds for non-symmetric and corresponding symmetrized states should remain the same. However, a state of  $N$  bosons needs not only to be permutationally invariant, but symmetric with respect to the interchange of any two particles, *i.e.*, it has to be possible to write it as a mixture of symmetric pure states fulfilling  $\hat{P}_{\pi} |\psi_{\text{S}}\rangle = |\psi_{\text{S}}\rangle$  for any permutation  $\pi$ . This is a much stronger requirement [43].

Consider, for example, the permutationally invariant state of  $N = 2$  particles  $\varrho_{\text{PI}} = [|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|]/2$ . Here  $|0\rangle$  and  $|1\rangle$  are eigenstates of the Pauli matrix  $\hat{\sigma}_z$  with eigenvalue  $+1$  and  $-1$ , respectively. This can be rewritten as  $\varrho_{\text{PI}} = [|\psi^{+}\rangle\langle\psi^{+}| + |\psi^{-}\rangle\langle\psi^{-}|]/2$ , where  $|\psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ . Hence  $\varrho_{\text{PI}}$  does not live on the symmetric subspace because it has an antisymmetric component  $|\psi^{-}\rangle$ . Since the state is separable,  $\xi \geq 1$  for any combination of the directions  $\mathbf{n}$  and  $\perp$ . Hence it does not allow for sub shot-noise phase estimation. Projecting  $\varrho_{\text{PI}}$  onto the symmetric subspace leads to  $\varrho_{\text{PI}} \rightarrow |\psi^{+}\rangle\langle\psi^{+}|$ . This state is known as a twin-Fock state of  $N = 2$  particles [38]. It is entangled [45] and allows for sub shot-noise phase estimation [38] even though it is not spin-squeezed because  $\langle\psi^{+}|\hat{J}_{\mathbf{n}}|\psi^{+}\rangle = 0$  for any  $\mathbf{n}$ . In Appendix B, we consider an additional example where a separable state is transformed into an entangled spin squeezed state by symmetrization.

This shows that symmetrization does not preserve neither entanglement nor spin squeezing. In general, symmetrization does not preserve the  $k$ -producibility class of a state of  $N$  particles. A  $k$ -producible state will generally be  $N$ -particle entangled after the symmetrization, and the bounds for a given  $k$  do not apply anymore.

### B. Generalizing the Sørensen-Mølmer criteria for indistinguishable particles

We assume that the collective spin transformations and measurements are performed on two energy levels of each atom, which we refer to as the internal degrees of freedom. The extension of the bounds (7) and (8) to indistinguishable particles is based on the inclusion of the atomic external degrees of freedom such as the spatial trap states. We thus consider operations of the form  $\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}$ , where  $\hat{A}_{\text{in}}$  acts on the internal degrees of freedom and  $\mathbb{1}_{\text{ex}}$  is the identity acting on the external degrees of freedom. The operator  $\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}$  must be permutationally invariant because we consider indistinguishable particles [44]. As mentioned above, this is the case for the collective spin operators  $\hat{A}_{\text{in}} = \hat{J}_{\mathbf{n}}$ .

The basic idea is that particles can be distinguished here by their external state. Therefore, the state needs to be symmetrized only with respect to all particles in the *same* external state, but not with respect to particles in *different* external states. This is true even though the operations introduced above do not resolve the external states [53].

Let us illustrate this with an example. We consider  $N = 2$  particles, labeled as 1 and 2, in two different external states, labeled as  $a$  and  $b$  ( $\langle a|b\rangle = 0$ ). Following Ref. [44], a general pure symmetric state can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_{12}\rangle_{\text{in}} \otimes |a_1 b_2\rangle_{\text{ex}} + |\psi_{21}\rangle_{\text{in}} \otimes |b_1 a_2\rangle_{\text{ex}}), \quad (21)$$

where  $|\psi_{12}\rangle_{\text{in}}$  is a general (not necessarily symmetric) internal state of the two particles,  $|\psi_{21}\rangle_{\text{in}} = \hat{P}_{\text{in}} |\psi_{12}\rangle_{\text{in}}$ ,  $\hat{P}_{\text{in}}$  permutes the particles, and  $|a_i b_j\rangle_{\text{ex}}$  is the external (e.g. spatial) wave function ( $i, j = 1, 2$ ,  $i \neq j$ ). The mean value of the operator  $\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}$  is

$$\langle\psi|\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}|\psi\rangle = \frac{\langle\psi_{12}|\hat{A}_{\text{in}}|\psi_{12}\rangle + \langle\psi_{21}|\hat{A}_{\text{in}}|\psi_{21}\rangle}{2}, \quad (22)$$

where the two terms in the sum are equal since  $\hat{A}_{\text{in}}$  is permutationally invariant:

$$\langle\psi_{21}|\hat{A}_{\text{in}}|\psi_{21}\rangle = \langle\psi_{12}|\hat{P}_{\text{in}}^{\dagger} \hat{A}_{\text{in}} \hat{P}_{\text{in}}|\psi_{12}\rangle = \langle\psi_{12}|\hat{A}_{\text{in}}|\psi_{12}\rangle. \quad (23)$$

We dropped the label “in” of  $|\psi_{12}\rangle_{\text{in}}$  for simplicity. The above equations show that  $|\psi_{12}\rangle$  is sufficient to describe the state of the two particles. In particular, non-symmetric states  $|\psi_{12}\rangle$  are allowed and the two particles can be formally treated as distinguishable.

The generalization to a system of  $N_{\gamma}$  particles in the external level  $\gamma$  (such that  $\sum_{\gamma} N_{\gamma} = N$ ) can be formulated as follows.

**Observation 3.** *The expectation value  $\langle\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}\rangle$  for any permutationally invariant operator  $\hat{A}_{\text{in}}$  with respect to a fully symmetric state  $|\psi\rangle$  with  $N_{\gamma}$  bosons in the external state  $\gamma$  is equal to the expectation value  $\langle\hat{A}_{\text{in}}\rangle$*



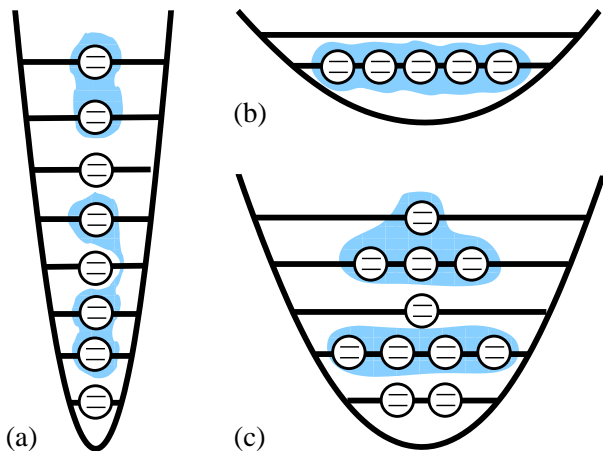


FIG. 1: Illustration of Observation 3 with particles of  $d_{\text{in}} = 2$  internal states. The light-gray structure (color online) indicates entanglement between particles. (a) Fully distinguishable particles ( $d_{\text{ex}} = N$ ). Depicted is a 3-particle entangled state of  $N = 8$  particles. Entanglement is present between the particles with energy level  $\gamma$  equal to 2, 3, and 5 (counting upwards from the lowest level with  $\gamma = 1$ ), and between particles 7 and 8. (b) Fully indistinguishable particles ( $d_{\text{ex}} = 1$ ). Depicted is a state which does not factorize. (c) Mixed situation ( $1 < d_{\text{ex}} < N$ ). Depicted is a 4-particle entangled state of  $N = 11$  particles in  $d_{\text{ex}} = 5$  external levels. There are two groups of 4 fully entangled particles: in the external level  $\gamma = 2$ , where the 4 particles are indistinguishable and in a non-factorisable state; and in levels 4 and 5. In the latter case, the group of particles in level  $\gamma = 4$  is distinguishable from the particle in level  $\gamma = 5$ . The two particles in the lowest level  $\gamma = 1$  are in a symmetric separable state of the form  $|\phi\rangle \otimes |\phi\rangle$ .

computed with respect to the corresponding internal state  $|\psi\rangle_{\text{in}}$ , which is symmetrized only with respect to the particles sharing the same state  $\gamma$ , for all  $\gamma$ .

This Observation is formulated more precisely in Appendix C below, where also the relationship of  $|\psi\rangle$  and  $|\psi\rangle_{\text{in}}$  is explained in detail.

Figure 1 illustrates several examples of  $N$  particles in  $d_{\text{in}} = 2$  internal and  $d_{\text{ex}}$  external modes. The usual situation employed in quantum information theory, where all particles are distinguishable ( $d_{\text{ex}} = N$ ), is shown in Fig. 1(a). For what concerns our discussion, this is formally equivalent to an array of separated wells, as in ion traps. The opposite situation of all particles occupying the same level ( $d_{\text{ex}} = 1$ ), is shown in Fig. 1(b). For indistinguishable particles, only two possibilities are allowed in this case: either all particles are in a separable (*i.e.*, product  $|\phi\rangle^{\otimes N}$ ) state, or all particles are entangled, due to the symmetrization [50–52]. As mentioned above, the  $k$ -particle entanglement criterion discussed in Sec. II does not apply in this case. The interesting intermediate situation is shown in Fig. 1(c). In this case several particles may occupy the same external state. As noticed in Observation 3, the symmetrization is necessary only

for particles that share the same external level  $\gamma$ . In this case, the  $N_\gamma$  particles may be only found in a fully entangled or fully separable state. However, entanglement is also possible between particles occupying different levels.

We can now extend the Sørensen-Mølmer bounds. A state can be considered as (effectively)  $k$ -producible if

$$|\psi\rangle_{\text{in}} = \otimes_{\alpha=1}^M |\psi_\alpha\rangle. \quad (24)$$

where  $|\psi_\alpha\rangle$  is state of  $N_\alpha \leq k$  particles ( $\sum_{\alpha=1}^M N_\alpha = N$ ) for all  $\alpha$ . The particles in the state  $|\psi_\alpha\rangle$  can occupy a single external state  $\gamma$  (in which case  $N_\alpha = N_\gamma$  and  $|\psi_\alpha\rangle = |\psi_\gamma\rangle$  is symmetric) or different external states  $\gamma \in I_\alpha$  (in which case  $N_\alpha = \sum_{\gamma \in I_\alpha} N_\gamma$  and  $|\psi_\alpha\rangle$  is not necessarily symmetric). As an example, the state schematically shown in Fig. 1(c) is 4-particle entangled.

With this notion, the Sørensen-Mølmer criteria can be applied in systems of indistinguishable particles as follows.

**Observation 4.** (i) For particles of spin  $j$ , if the spin squeezing parameter violates Eq. (7) for a given  $k$ , then the input state cannot be written as a mixture of effectively  $k$ -producible states of Eq. (24). (ii) For particles of spin  $\frac{1}{2}$ , if the spin squeezing parameter violates Eq. (7) for a given  $k$ , then the input state allows for a smaller phase uncertainty than the smallest one achievable with a mixture of effectively  $k$ -producible states of Eq. (24).

In both cases, effective  $(k + 1)$ -particle entanglement is proven by a violation of the criteria. These notions directly generalize to systems of a fluctuating number of particles as in Section II.

### C. Cold atoms

Observation 3 is also useful in the context of entanglement detection with generalized spin squeezing inequalities (SSI) [3, 6] in cold atomic clouds. Usually the atomic ensembles are not ultracold, and can be assumed to be in a thermal state externally. We estimate the population of the trap levels using the statistics of an ideal Bose gas taking the parameters from a typical experiment with a cigar-shaped configuration [54, 55]:  $\omega_z = 2$ ,  $\omega_\perp = 1000$  (trap frequencies in units of  $2\pi\text{s}^{-1}$ ),  $T = 30\mu\text{K}$ ,  $N = 5 \cdot 10^5$ . The chemical potential  $\mu$  is defined implicitly by the relation  $N = \sum_k \langle n_k \rangle$ , where [56]

$$\langle n_k \rangle = \sum_n \left[ \frac{e^{-\beta n(E_k - \mu)}}{\sum_{n'} e^{-\beta n'(E_k - \mu)}} \right] n = \frac{1}{e^{\beta(E_k - \mu)} - 1} \quad (25)$$

is the average population of the level  $k$ ,  $\beta = 1/k_B T$  with the Boltzmann constant  $k_B$ , and  $E_k$  is the energy of level  $k$ . We approximate this by the energy levels of a 3d harmonic oscillator with the given trap frequencies.

The largest average population is obtained for the ground state with energy  $E_0 = \hbar(\omega_z/2 + \omega_\perp)$ . We obtain  $\beta\mu \approx -12.41$ , which leads to  $\langle n_0 \rangle \approx 4.1 \cdot 10^{-6}$ . Further,

the probability of having  $n_k$  particles in the level  $k$ , given by the expression in square brackets in Eq. (25), decreases exponentially with  $n_k$ . The ratio  $p_{n_k+1}/p_{n_k} = e^{-\beta(E_k-\mu)}$  is the largest for  $k = 0$ , where it is equal to  $4.1 \cdot 10^{-6}$ . Therefore, it can be assumed that at most one particle occupies each level. Since only internal quantities are used in the generalized SSI, one can therefore treat the particles as distinguishable by using Obs. 3. This situation corresponds to the one depicted in Fig. 1(a).

#### D. Bose-Einstein condensates

In Bose-Einstein condensates particles share the same external state, as schematically illustrated in Fig. 1(b). In this case, it is not possible to directly apply the bounds (7) and (8): particle entanglement is either absent or maximal, due to symmetrization [50–52].

However, we recall that Eqs. (7) and (8) are sufficient conditions for entanglement and, as mentioned in the introduction, for spin  $j = \frac{1}{2}$  particles, they are strongly related to the usefulness of the entangled state for parameter estimation. Therefore, a measurement of the spin squeezing bounds in BECs might still be consistent with  $k < N$ , even if, by some other means, it is possible to show that *all* particles share the same external state, as in Ref. [11]. The outcome  $k < N$  should be interpreted either as due to noise or by saying that all particles are entangled but the state is only partially useful for parameter estimation. It is known, indeed, that there are symmetric states which are fully  $N$ -particle entangled, but that are not spin squeezed and do not allow for sub shot-noise phase estimation [45].

Finally, one might think that, by making the BEC cloud very dilute, it is possible to effectively distinguish the particles and thus use the spin squeezing bounds (7) and (8). In Appendix D we show that this is not the case.

#### IV. CONCLUSIONS

The spin squeezing criteria introduced by Sørensen and Mølmer for  $N$  distinguishable particles in Ref. [10] are a powerful and experimentally feasible method to detect  $k$ -particle entanglement, also referred to as entanglement depth  $k$ . However, most of the spin squeezing experiments are performed with a fluctuating number of particles and, as in the case of BEC, these particles are indistinguishable. To fill this gap between theory and experiment, we have extended, in the first part of this article, the Sørensen and Mølmer criteria to systems with a fluctuating number of particles. We have also shown how other spin squeezing inequalities [20, 21] can be generalized to this situation. In the second part of the paper, we discussed the conceptual problems that occur when the individual particles are indistinguishable. In this case, effective  $k$ -particle entanglement can be defined only by

making use of additional degrees of freedom of the atoms. The spin squeezing bounds of Ref. [10] can then be interpreted as conditions of such effective  $k$ -particle entanglement. Our results make it possible to apply the bounds of Ref. [10] in spin squeezing experiments with cold atoms and Bose-Einstein condensates.

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#### APPENDIX

##### A. Proof of Observation 1

The proof follows the lines of the proof of Eq. (7) for fixed  $N$  [10] using methods developed for non-fixed  $N$  in Ref. [13]. We want to compute a lower bound on the variance of  $\hat{J}_\perp$  for all  $k$ -producible states. Since the variance is concave in the state, its minimum value is reached by pure states of the form  $|\psi_{k\text{-prod}}\rangle = \sum_N \sqrt{Q_N} |\psi_{k\text{-prod}}^{(N)}\rangle$  [57], where  $\sqrt{Q_N}$  are real numbers with  $\sum_N Q_N = 1$  and  $|\psi_{k\text{-prod}}^{(N)}\rangle = \bigotimes_{\alpha=1}^{M_N} |\psi_\alpha^{(N_\alpha)}\rangle$  is a  $k$ -producible state of  $N$  particles [cf. Eq. (3)]. Using Eq. (9) we can write  $(\Delta \hat{J}_\perp)^2 \geq \sum_N Q_N (\Delta \hat{J}_\perp^{(N)})^2$ . In addition, we note that, due to the product structure of the states  $|\psi_{k\text{-prod}}^{(N)}\rangle$  and since  $\hat{J}_\perp^{(N)}$  is the sum of operators acting on fixed  $N_\alpha$  subspaces,  $\hat{J}_\perp^{(N)} = \sum_{\alpha=1}^{M_N} \hat{J}_\perp^{(N_\alpha)}$ , we have  $(\Delta \hat{J}_\perp^{(N)})^2 = \sum_{\alpha=1}^{M_N} (\Delta \hat{J}_\perp^{(N_\alpha)})^2$ . Therefore, the variance of  $\hat{J}_\perp$  for  $k$ -producible states is bounded by

$$(\Delta \hat{J}_\perp)^2 \geq \sum_N Q_N \sum_{\alpha=1}^{M_N} (\Delta \hat{J}_\perp^{(N_\alpha)})^2, \quad (26)$$

where the operator on the right hand side acts on the  $N_\alpha$  particles in the state  $|\psi_\alpha^{(N_\alpha)}\rangle$ . Note that we did not attach an index  $N$  to  $N_\alpha$  in order to simplify the notation.

Now we have to find the minimal bound for the variances  $(\Delta \hat{J}_\perp^{(N_\alpha)})^2$  for every  $N$ , given the mean value  $\langle \hat{J}_\perp^{(N_\alpha)} \rangle$ . If we consider  $N_\alpha$  spin- $j$  particles, the total spin  $j_\alpha$  can range from 0 (if  $N_\alpha$  is even) or  $1/2$  (if  $N_\alpha$  is odd) up to  $N_\alpha j$ . We show that for any  $\langle \hat{J}_\perp^{(N_\alpha)} \rangle$ , the smallest bound is reached by choosing the largest total spin possible by using that  $(\Delta \hat{J}_\perp^{(N_\alpha)})^2 \geq j_\alpha F_{j_\alpha}(\langle \hat{J}_\perp^{(N_\alpha)} \rangle / j_\alpha)$

for states with a fixed spin  $j_\alpha$ , cf. Eq. (6). The ingredients needed for this, which have been proven in Ref. [10], are: (i) the functions  $F_{j_\alpha}$  are convex, *i.e.*,  $F_{j_\alpha}(aX+bY) \leq aF_{j_\alpha}(X)+bF_{j_\alpha}(Y)$  for all  $j_\alpha$  and  $a, b \geq 0$  with  $a+b=1$ , (ii)  $F_{j_\alpha}(0) = 0$  for all  $j_\alpha$ , and (iii) that  $F_{j_\alpha}(X) \leq F_{j'_\alpha}(X)$  if  $j_\alpha \geq j'_\alpha$ . By using the inequality (i) with  $a = \frac{j'_\alpha}{j_\alpha}$ ,  $Y = 0$  and  $b = 1 - a$ , and the property (ii), we have  $F_{j_\alpha}(\frac{j'_\alpha}{j_\alpha}X) \leq j'_\alpha F_{j_\alpha}(X)/j_\alpha$ . Multiplying by  $j_\alpha$  both terms and using (iii), we arrive at  $j_\alpha F_{j_\alpha}(\frac{j'_\alpha}{j_\alpha}X) \leq j'_\alpha F_{j'_\alpha}(X)$  if  $j'_\alpha \leq j_\alpha$ . Finally, taking  $X = \frac{\langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle}{j'_\alpha}$ , we have

$$j_\alpha F_{j_\alpha}\left(\frac{\langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle}{j_\alpha}\right) \leq j'_\alpha F_{j'_\alpha}\left(\frac{\langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle}{j'_\alpha}\right) \quad (27)$$

if  $j_\alpha \geq j'_\alpha$ . Let us now consider a superposition  $|\psi\rangle = c_{j_\alpha}|\psi_{j_\alpha}\rangle + c_{j'_\alpha}|\psi_{j'_\alpha}\rangle$  of states with a different fixed spin  $j_\alpha \geq j'_\alpha$ . Since the spin operator  $\hat{J}_\perp$  does not couple the states of different total spin  $j_\alpha$  and  $j'_\alpha$ , its variance with respect to  $|\psi\rangle$  is equal to the variance with respect to the mixture  $\rho = |c_{j_\alpha}|^2|\psi_{j_\alpha}\rangle\langle\psi_{j_\alpha}| + |c_{j'_\alpha}|^2|\psi_{j'_\alpha}\rangle\langle\psi_{j'_\alpha}|$ . Using the concavity of the variance, we obtain that

$$\begin{aligned} (\Delta \hat{J}_\perp)_{|\psi\rangle}^2 &\geq |c_{j_\alpha}|^2(\Delta \hat{J}_\perp)_{|\psi_{j_\alpha}\rangle}^2 + |c_{j'_\alpha}|^2(\Delta \hat{J}_\perp)_{|\psi_{j'_\alpha}\rangle}^2 \\ &\geq |c_{j_\alpha}|^2 F_{j_\alpha}(\langle \hat{J}_\mathbf{n} \rangle_{|\psi_{j_\alpha}\rangle} / j_\alpha) + |c_{j'_\alpha}|^2 F_{j'_\alpha}(\langle \hat{J}_\mathbf{n} \rangle_{|\psi_{j'_\alpha}\rangle} / j'_\alpha) \\ &\geq F_{j_\alpha}(\langle \hat{J}_\mathbf{n} \rangle_{|\psi\rangle} / j_\alpha), \end{aligned}$$

where we have used Eq. (27) and  $|c_{j_\alpha}|^2 + |c_{j'_\alpha}|^2 = 1$ .

Taking the maximum value of  $j_\alpha$  (*i.e.*,  $j_\alpha = N_\alpha j$ ) we arrive at

$$(\Delta \hat{J}_\perp^{(N_\alpha)})^2 \geq N_\alpha j F_{N_\alpha j} \left( \frac{\langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle}{N_\alpha j} \right) \geq N_\alpha j F_{kj} \left( \frac{\langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle}{N_\alpha j} \right), \quad (28)$$

where the second inequality is due to (iii) and  $N_\alpha \leq k$  for  $k$ -producible states. Since the function  $F_{j_\alpha}(X)$  is convex in  $X$ , we can now apply Jensen's inequality [59] to the last term in Eq. (28). We obtain that

$$\sum_{\alpha=1}^{M_N} (\Delta \hat{J}_\perp^{(N_\alpha)})^2 \geq N j F_{kj} \left( \frac{\langle \hat{J}_\mathbf{n}^{(N)} \rangle}{N j} \right), \quad (29)$$

where  $\langle \hat{J}_\mathbf{n}^{(N)} \rangle = \sum_\alpha \langle \hat{J}_\mathbf{n}^{(N_\alpha)} \rangle$  for any  $N$ . Finally, by combining Eqs. (26) and (29), and using again Jensen's inequality [59], we have

$$(\Delta \hat{J}_\perp)^2 \geq \sum_N Q_N N j F_{kj} \left( \frac{\langle \hat{J}_\mathbf{n}^{(N)} \rangle}{N j} \right) \geq \langle \hat{N} \rangle j F_{kj} \left( \frac{\langle \hat{J}_\mathbf{n} \rangle}{\langle \hat{N} \rangle j} \right),$$

where  $\langle \hat{N} \rangle = \sum_N Q_N N$  and  $\langle \hat{J}_\mathbf{n} \rangle = \sum_N Q_N \langle \hat{J}_\mathbf{n}^{(N)} \rangle$ . This proves Obs. 1 [cf. Eq. (8)].  $\square$

## B. Example: symmetrization creates spin squeezing

Let us consider the state

$$|\psi_\alpha\rangle = \sqrt{\alpha}|11\rangle + \sqrt{1-\alpha}|01\rangle = (\sqrt{\alpha}|1\rangle + \sqrt{1-\alpha}|0\rangle) \otimes |1\rangle \quad (30)$$

which is clearly separable. Therefore, it has a spin squeezing parameter  $\xi^2 \geq 1$  for any  $\alpha$  and any combination of the directions  $\mathbf{n}$  and  $\perp$ , *i.e.*, it is not spin squeezed. In particular, for the directions  $\mathbf{n} = \mathbf{z}$  and  $\perp = \mathbf{x}$  it is given by [cf. Eq. (1)]

$$\xi_\alpha^2 = \frac{1}{\alpha^2} - \frac{2}{\alpha} + 2 \geq 1. \quad (31)$$

The corresponding state which could be realized in the scenario of indistinguishable bosons is obtained by symmetrizing (and normalizing) the state  $|\psi_\alpha\rangle$ , leading to

$$|\psi_\alpha^S\rangle = \sqrt{\beta}|11\rangle + \sqrt{1-\beta} \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad (32)$$

where  $\beta = \frac{2\alpha}{1+\alpha}$ . For this state, the spin squeezing parameter for the same particular directions  $\mathbf{n} = \mathbf{z}$  and  $\perp = \mathbf{x}$  is given by [cf. Eq. (1)]

$$\xi_\beta^2 = \frac{2}{\alpha^2} - \frac{5}{\alpha} + 4. \quad (33)$$

This is smaller than the critical value 1 for a larger range of parameters. The minimum is reached at  $\xi_\beta^2 = 7/8$  for  $\beta = 4/5$ , which corresponds to  $\alpha = 2/3$ . Therefore, symmetrization does not preserve neither entanglement nor spin squeezing. In order to illustrate the results, the two curves are plotted in Fig. 2.

## C. Proof of Observation 3

Let us first introduce the formalism used in the proof. Recall that we are considering here  $N$  particles with  $d_{\text{in}}$  ( $d_{\text{ex}}$ ) internal (external) degrees of freedom, labeled by  $i = 1, \dots, d_{\text{in}}$  ( $\gamma = 1, \dots, d_{\text{ex}}$ ). Referring to Fig. (1), we can think of the external states as the energy levels of a spatial trap. Each level  $\gamma$  contains  $N_\gamma$  particles in the state  $|\gamma\rangle_{\text{ex}}$ . These particles can be in different internal states  $|i\rangle_{\text{in}}$ . In general, we have  $N_{i,\gamma}$  particles in the state  $|i, \gamma\rangle \equiv |i\rangle_{\text{in}} \otimes |\gamma\rangle_{\text{ex}}$  (to simplify the notation, we remove here the tensor product sign and penders “in” and “ex”). We also introduce a vector  $\mathbf{N}_{\text{ex}} = (N_1, N_2, \dots, N_{d_{\text{ex}}})$  giving the occupation numbers of each external state and a vector  $\mathbf{N}_\gamma = (N_{1,\gamma}, N_{2,\gamma}, \dots, N_{d_{\text{in}},\gamma})$  with occupations of the internal states for a fixed external level  $\gamma$ . Here, the relations  $\sum_{\gamma=1}^{d_{\text{ex}}} N_\gamma = N$  and  $\sum_{i=1}^{d_{\text{in}}} N_{i,\gamma} = N_\gamma$  hold.

Let us consider a specific example for  $d_{\text{in}} = d_{\text{ex}} = 2$  and  $N = 3$ . Choosing  $N_{1,1} = 1$ ,  $N_{2,1} = 0$ ,  $N_{1,2} = 2$  and  $N_{2,2} = 0$ , we obtain  $\mathbf{N}_{\text{ex}} = (1, 2)$ ,  $\mathbf{N}_1 = (1, 0)$  and  $\mathbf{N}_2 = (2, 0)$ . The (non-symmetric) state is

$$\otimes_{\gamma=1}^2 \otimes_{i=1}^2 |i, \gamma\rangle^{\otimes N_{i,\gamma}} = |1, 1\rangle \otimes |1, 2\rangle \otimes |1, 2\rangle. \quad (34)$$

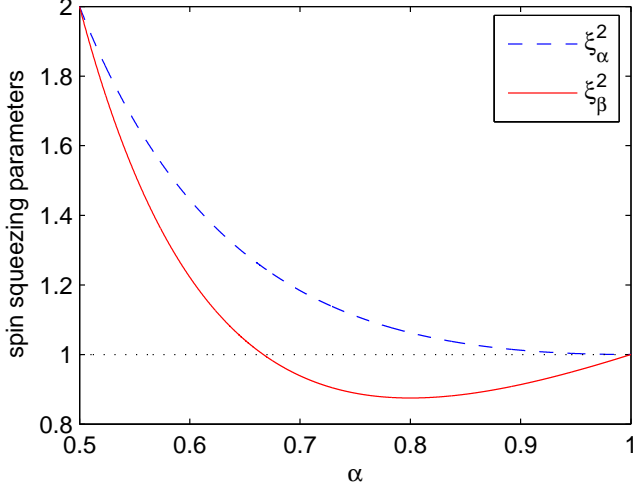


FIG. 2: Plot of the spin squeezing parameters  $\xi_\alpha^2$  [dashed line, Eq. (31)] and  $\xi_\beta^2$  [solid line, Eq. (33)]. While  $\xi_\alpha^2$  never goes below the value 1 [dotted line],  $\xi_\beta^2$  is below 1 for a large interval.

Finally,  $\{\mathbf{N}_\gamma\}_\gamma$  is the complete set of occupation numbers  $\mathbf{N}_\gamma$  for all the  $\gamma$  levels. The corresponding symmetrized states with occupation numbers  $\mathbf{N}_\gamma$  is given by

$$|D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}}\rangle \equiv \frac{1}{\sqrt{\mathcal{N}}} \sum_{\pi} \hat{P}_{\pi} [\otimes_{\gamma=1}^{d_{\text{ex}}} \otimes_{i=1}^{d_{\text{in}}} |i, \gamma\rangle^{\otimes N_{i,\gamma}}], \quad (35)$$

where  $\hat{P}_{\pi}$  is a representation of the permutation  $\pi$ , and the sum runs through all *distinct* permutations, the number of which is  $\mathcal{N} \equiv \binom{N}{\{\mathbf{N}_\gamma\}} \equiv \frac{N!}{\prod_{i,\gamma} N_{i,\gamma}!}$ . The states  $|D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}}\rangle$  form a basis which is the analogous to a Fock state basis in second quantization.

The non-symmetric state from the example above [cf. Eq. (34)] becomes

$$|D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}}\rangle = \frac{1}{\sqrt{3}} (|1,1\rangle \otimes |1,2\rangle \otimes |1,2\rangle + |1,2\rangle \otimes |1,1\rangle \otimes |1,2\rangle + |1,2\rangle \otimes |1,2\rangle \otimes |1,1\rangle).$$

We will use the label  $D$  in general for Fock states with a fixed occupation in internal and external levels in first quantization. In particular, we employ symmetric states with  $N_\gamma$  particles in the single external level  $\gamma$ ,

$$|D_{N_\gamma}^{\mathbf{N}_\gamma}\rangle \equiv |I_{N_\gamma}^{\mathbf{N}_\gamma}\rangle \otimes |\gamma\rangle^{\otimes N_\gamma}, \quad (36)$$

where

$$|I_{N_\gamma}^{\mathbf{N}_\gamma}\rangle \equiv \frac{1}{\sqrt{\mathcal{N}_\gamma}} \sum_{\pi} \hat{P}_{\pi} [\otimes_i |i\rangle_{\gamma}^{\otimes N_{i,\gamma}}], \quad (37)$$

and  $\mathcal{N}_\gamma \equiv \binom{N_\gamma}{\mathbf{N}_\gamma} \equiv \frac{N_\gamma!}{\prod_{i,\gamma} N_{i,\gamma}!}$  is the number of distinct permutations  $\pi$ . We attach the label  $\gamma$  to  $|i\rangle$  in order to keep

track of the external level  $\gamma$  the particle is in. This will be important below.

With these definitions, we can reformulate Observation 3 in technical terms.

**Observation 3.** For any permutationally invariant operator  $\hat{A}_{\text{in}}$  acting on the internal degree of freedom, and for a symmetric state  $|\Psi_{\text{S}}^{\mathbf{N}_{\text{ex}}}\rangle = \sum_{\{\mathbf{N}_\gamma\}} c_{\{\mathbf{N}_\gamma\}} |D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}}\rangle$  with a fixed occupation vector  $\mathbf{N}_{\text{ex}}$ ,

$$\langle \Psi_{\text{S}}^{\mathbf{N}_{\text{ex}}} | \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}} | \Psi_{\text{S}}^{\mathbf{N}_{\text{ex}}} \rangle = \langle \Psi_{\text{in}}^{\mathbf{N}_{\text{ex}}} | \hat{A}_{\text{in}} | \Psi_{\text{in}}^{\mathbf{N}_{\text{ex}}} \rangle \quad (38)$$

holds, where  $|\Psi_{\text{in}}^{\mathbf{N}_{\text{ex}}}\rangle = \sum_{\{\mathbf{N}_\gamma\}} c_{\{\mathbf{N}_\gamma\}} [\otimes_{\gamma} |I_{N_\gamma}^{\mathbf{N}_\gamma}\rangle]$ , and  $|I_{N_\gamma}^{\mathbf{N}_\gamma}\rangle$  is a symmetric internal state as defined in Eq. (37).

*Proof.* By inserting the definitions of  $|\Psi_{\text{S}}^{\mathbf{N}_{\text{ex}}}\rangle$  and  $|\Psi_{\text{in}}^{\mathbf{N}_{\text{ex}}}\rangle$  it is easy to see that Eq. (38) holds if

$$\langle D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}} | \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}} | D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}'_\gamma\}} \rangle = [\otimes_{\gamma} \langle I_{N_\gamma}^{\mathbf{N}_\gamma} |] \hat{A}_{\text{in}} [\otimes_{\gamma'} |I_{N_{\gamma'}}^{\mathbf{N}'_{\gamma'}}\rangle] \quad (39)$$

is true for all  $\{\mathbf{N}_\gamma\}$  and  $\{\mathbf{N}'_\gamma\}$  with the same  $\mathbf{N}_{\text{ex}}$ . We will show now that this is the case. We insert into the left hand side of Eq. (39) the definition of the states  $|D_{\mathbf{N}_{\text{ex}}}^{\{\mathbf{N}_\gamma\}}\rangle$  [cf. Eq. (35)], which leads to

$$\sum_{\pi, \pi'} \left[ \frac{\hat{P}_{\pi}}{\sqrt{\mathcal{N}}} \otimes_{\gamma, i} |i, \gamma\rangle^{\otimes N_{i,\gamma}} \right]^{\dagger} \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}} \left[ \frac{\hat{P}_{\pi'}}{\sqrt{\mathcal{N}'}} \otimes_{\gamma', i'} |i', \gamma'\rangle^{\otimes N_{i',\gamma'}} \right]. \quad (40)$$

As before, we consider the sum of distinct permutations only. Due to the identity  $\mathbb{1}_{\text{ex}}$  on the external states the terms in the sum will vanish unless the  $N_\gamma$  particles in level  $\gamma$  are on the same positions in the permutations on both sides of  $\hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}$ , since  $\langle i, \gamma | \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}} | i', \gamma' \rangle = \langle i | \hat{A}_{\text{in}} | i' \rangle \delta_{\gamma, \gamma'}$ . Therefore, we can rewrite expression (40) as

$$\sum_{\pi, \tilde{\pi}, \tilde{\pi}'} \left[ \frac{\hat{P}_{\pi} \hat{P}_{\tilde{\pi}}}{\sqrt{\mathcal{N}}} \otimes_{\gamma, i} |i\rangle_{\gamma}^{\otimes N_{i,\gamma}} \right]^{\dagger} \hat{A}_{\text{in}} \left[ \frac{\hat{P}_{\pi} \hat{P}_{\tilde{\pi}'}}{\sqrt{\mathcal{N}'}} \otimes_{\gamma', i'} |i'\rangle_{\gamma'}^{\otimes N_{i',\gamma'}} \right]. \quad (41)$$

Here the permutations  $\tilde{\pi}$  and  $\tilde{\pi}'$  permute particles stemming from the same external state  $\gamma$ , and  $\pi$  permutes particles with a different  $\gamma$ . Note that for simplicity we use the same operators  $\hat{P}_{\tilde{\pi}}$  to represent a permutation  $\tilde{\pi}$  of the  $N$  particles even though now the state space of each particle is reduced to the internal states. In order to clarify the notation we employed, we note that for the example considered in Eq. (34), the reduced state would be

$$\otimes_{\gamma, i} |i\rangle_{\gamma}^{\otimes N_{i,\gamma}} = |1\rangle_1 \otimes |1\rangle_2^{\otimes 2} = |1\rangle_1 \otimes |1\rangle_2 \otimes |1\rangle_2.$$

Since  $\hat{A}_{\text{in}}$  is permutationally invariant, we have that  $\hat{P}_{\pi}^{\dagger} \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}} \hat{P}_{\pi} = \hat{A}_{\text{in}} \otimes \mathbb{1}_{\text{ex}}$ . Hence in the sum over  $\pi$  each term contributes equally, and the sum can be replaced by the number of distinct permutations  $\mathcal{N}_{\text{ex}} \equiv \binom{N}{\mathbf{N}_{\text{ex}}} \equiv \frac{N!}{\prod_{\gamma} N_{\gamma}!}$  in expression (41).



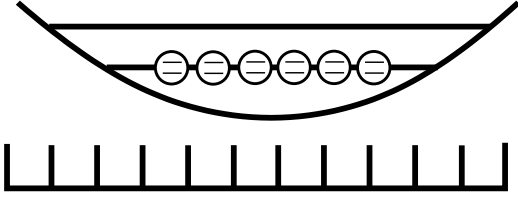


FIG. 3: Initially all particles are in the lowest energy level of the trap. The cloud is then released. If it is diluted enough along the horizontal direction, it is likely that at most one particle falls into each of the boxes, which represent the single detectors.

We observe that because the permutations  $\tilde{\pi}$  only permute particles with the same  $\gamma$ , one can rewrite  $\sum_{\tilde{\pi}} \hat{P}_{\tilde{\pi}} = \Pi_{\gamma}(\sum_{\pi_{\gamma}} \hat{P}_{\pi_{\gamma}})$  of permutations  $\pi_{\gamma}$  which permute particles in the level  $\gamma$ . This leads to

$$\begin{aligned} \sum_{\tilde{\pi}} \hat{P}_{\tilde{\pi}} [\otimes_{\gamma,i} |i\rangle_{\gamma}^{\otimes N_{i,\gamma}}] &= \otimes_{\gamma} \sum_{\pi_{\gamma}} \hat{P}_{\pi_{\gamma}} [\otimes_i |i\rangle_{\gamma}^{\otimes N_{i,\gamma}}] \\ &= \sqrt{\Pi_{\gamma} \mathcal{N}_{\gamma}} [\otimes_{\gamma} |I_{N_{\gamma}}^{\mathbf{N}_{\gamma}}\rangle], \end{aligned}$$

cf. Eq. (37). We arrive at

$$\sqrt{\frac{\mathcal{N}_{\text{ex}}^2 \Pi_{\gamma} \mathcal{N}_{\gamma} \Pi_{\gamma'} \mathcal{N}_{\gamma'}}{\mathcal{N}_{\gamma} \mathcal{N}_{\gamma'}}} [\otimes_{\gamma} \langle I_{N_{\gamma}}^{\mathbf{N}_{\gamma}} |] \hat{A}_{\text{in}} [\otimes_{\gamma'} |I_{N_{\gamma'}}^{\mathbf{N}_{\gamma'}}\rangle].$$

One can directly check that the prefactor is equal to 1. Therefore, condition Eq. (39) is fulfilled.  $\square$

#### D. Dilute cloud argument

One may think that, after preparing the Bose-Einstein condensate atoms in the ground state of a confining trap, it is possible to apply the spin squeezing inequalities (7) and (8) by simply releasing the trap, letting the cloud expand and fall onto a grid of small detectors capable of measuring the internal state of a single atom. If the cloud is dilute enough, it is very likely that at most a single atom enters each detector, thus making the atoms distinguishable. This would make it possible to apply the Sørensen-Mølmer bounds. The situation is illustrated in Fig. 3. We show here that this argument, which is often encountered in discussions, does resolve the problem.

Let us assume that, before releasing the atoms from the trap, their state is of the form

$$|\Psi\rangle = |\psi_{\text{S}}\rangle_{\text{in}} \otimes |0\rangle_{\text{ex}}^{\otimes N}, \quad (42)$$

where  $|\psi_{\text{S}}\rangle_{\text{in}}$  is a symmetric total internal state and each atom is in the ground state of the trap  $|0\rangle_{\text{ex}}$ . Here “ex” (“in”) indicates the external (internal) degree of freedom as in Section III B. We assume here that all atoms share the same spatial wave function, which can thus be factorized. If interactions can be neglected during free fall, then

only the spatial state of each atom changes, leaving the internal state symmetric. By waiting long enough, the single-particle spatial wave function becomes so spread that the probability to detect two atoms at the same spatial detector is negligible.

Let us assume for simplicity that the atoms are trapped and detected state-insensitively first, such that at most one atom is detected in each site. A problem is that in each shot, different sites will be occupied. This might still be considered as a minor problem. In a one-dimensional trap, for instance, it could be resolved by identifying particle “1” with the leftmost trap, particle “2” with the particle right from particle “1”, and so on. Alternatively, one could postselect on events where always the same  $N$  sites are occupied.

In general, the position measurement makes the state effectively distinguishable. Let us illustrate the situation with an example for  $N = 2$  particles, labeled as 1 and 2, in two different sites labeled by  $a$  and  $b$ . As in Section III B, we consider a general pure symmetric state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_{12}\rangle_{\text{in}} \otimes |a_1 b_2\rangle_{\text{ex}} + |\psi_{21}\rangle_{\text{in}} \otimes |b_1 a_2\rangle_{\text{ex}}), \quad (43)$$

with the same definitions as in Eq. (21). An operator  $\hat{M}_a$  acting on the internal state of the particle in site  $a$  can be written as

$$\begin{aligned} \hat{M}_a &= (\hat{A}_1 \otimes \mathbb{1}_2)_{\text{in}} \otimes (\hat{m}_1^a \otimes \mathbb{1}_2)_{\text{ex}} \\ &\quad + (\mathbb{1}_1 \otimes \hat{A}_2)_{\text{in}} \otimes (\mathbb{1}_1 \otimes \hat{m}_2^a)_{\text{ex}}, \end{aligned}$$

where  $\hat{m}^a|a\rangle = |a\rangle$  and  $\hat{m}^a|b\rangle = 0$  since we measure locally at site  $a$ .  $\hat{M}_a$  has to be permutationally invariant with respect to the interchange of the particle labels since the particles are indistinguishable [44]. The expectation value with respect to the state (43) is

$$\langle\psi|\hat{M}_a|\psi\rangle = \frac{1}{2} [\langle\psi_{12}|\hat{A}_1 \otimes \mathbb{1}_2|\psi_{12}\rangle + \langle\psi_{21}|\mathbb{1}_1 \otimes \hat{A}_2|\psi_{21}\rangle], \quad (44)$$

where the two terms are equal since

$$\begin{aligned} \langle\psi_{21}|\mathbb{1}_1 \otimes \hat{A}_2|\psi_{21}\rangle &= \langle\psi_{12}|\hat{P}_{\text{in}}^{\dagger} (\mathbb{1}_1 \otimes \hat{A}_2) \hat{P}_{\text{in}}|\psi_{12}\rangle \\ &= \langle\psi_{12}|\hat{A}_1 \otimes \mathbb{1}_2|\psi_{12}\rangle. \end{aligned}$$

We dropped the label “in” of  $|\psi_{12}\rangle_{\text{in}}$  for simplicity. An analogous result is obtained when considering an operator acting on the internal state of the particle on site  $b$ . Since only such operators are measured in the usual scenario, we can identify particle 1 with site  $a$  and particle 2 with site  $b$ , and  $|\psi_{12}\rangle$  is sufficient to describe the state of the two particles. This is a state of two distinguishable particles.

However, since the measurement acts only on the external degrees of freedom, the product structure between the internal and the external degrees of freedom in Eq. (42) is preserved. It is then evident that the internal state remains fully symmetric even after the position measurement, since this affects the external state only.

Hence making the state distinguishable effectively after the state transformation in the measurement does not make it possible to leave the restricted class of symmetric states: in the interferometric situation we considered above, when only the sites  $a$  and  $b$  are occupied, then we arrive at the state of Eq. (43), but with a symmetric

internal state  $|\psi_{12}\rangle$ . Therefore, the effective state of the distinguishable particles is symmetric. This example, which can be directly generalized to  $N$  particles, shows that simply making the cloud dilute does not make it possible to apply the spin squeezing bounds discussed in this paper.

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$$\phi\left(\frac{\sum_i a_i x_i}{\sum_i a_i}\right) \leq \frac{\sum_i a_i \phi(x_i)}{\sum_i a_i}, \quad (45)$$

holds for any convex function  $\phi$  and positive weights  $a_i$  [58].